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Closed forms for European options in a local volatility model

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Abstract Because of its very general formulation, the local volatility model does not have an analytical solution for European options. In this article, we present a new methodology to derive closed form solutions for the price of any European options. The formula results from an asymptotic expansion, terms of which are Black-Scholes price and related Greeks. The accuracy of the formula depends on the payoff smoothness and it converges with very few terms.

Keywords local volatility model · European options · asymptotic expansion · Malliavin calculus · small diffusion process · CEV model

JEL Classification: G13

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1 Introduction

The local volatility model, introduced by Dupire [Dup94], Rubinstein [Rub94] and Derman Khani [ED94], has the main advantage of fitting all call and put option prices. However, in contrast to the seminal Black-Scholes model, this model has no more closed form solution for general European options. This comes from the very general form of the local volatility function. Only in a few cases this model admits closed formulas, as explained in [ACCL01]. In the special case of a separable local volatility function written as the product of two independent functions of time and underlying, $\sigma_{loc}(t,f) = \alpha(t)A(f)$, one can derive an asymptotic expansion for the price of vanilla options (call, put) using singular perturbation techniques as explained in [HW99].

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Another type of asymptotic expansion can be also derived from an expansion of the heat kernel as shown in [Lab05]. However, for the general case, there is no methodology so far. This paper tackles precisely this challenge.
The overall idea is to do an asymptotic expansion directly on the diffusion using Malliavin calculus. We will consider a local volatility model, in which the underlying asset is classically related to the diffusion process

$$dX_t = \sigma(t,X_t)dW_t + \mu(t,X_t)dt, \quad X_0 = x_0. \tag{1.1}$$

Typically, in the following, X stands for the log-price of the underlying asset[1]. $\sigma(t,X_t)$ is the volatility term whereas $\mu(t,X_t)$ is the drift term. Our aim is to give an analytical accurate[2] approximation of any European option, written as the expected value under the risk neutral probability measure of a payoff function h evaluated at the maturity time T:

$$\mathbb{E}(h(X_T)) \tag{1.2}$$

where $\mathbb{E}$ stands for the standard expectation operator. To accomplish this, we introduce a parametrized process given by:

$$dX_t^\varepsilon = \varepsilon(\sigma(t,X_t^\varepsilon)dW_t + \mu(t,X_t^\varepsilon)dt), X_0^\varepsilon = x_0, \tag{1.3}$$

where the parameter ε lies in the range $[0,1]$. Obviously, this parametrized process is equal to the initial one for $\varepsilon = 1$. Remarkably, it is much easier to calculate the price (1.2) as an expansion formula with respect to ε. Once we have derived all the terms of the expansion, we see that the price of the European option is obtained by taking $\varepsilon = 1$.
Compared to standard expansion methods, the accuracy of this expansion is not related to the perturbation parameter ε. Indeed, the limit value $\varepsilon = 1$ is not small at all. This is a significant difference compared to singular perturbation techniques. Our expansion is just a way to derive convenient closed form solution. This asymptotic expansion is achieved using the infinite dimensional analysis of Malliavin calculus. A key feature of our approach is that we can provide explicit formulas for the terms at any order and explicit upper bounds of the errors, for general forms of the drift term μ and the volatility term σ. The derivation of expansion terms at any order completes for pure diffusion some earlier work done in [BGM08].
In practice, we compute a limited number of terms. The main term is the price in a suitable Black-Scholes model, while the other terms are a weighted summation of sensitivities (Greeks). These terms are straightforward to evaluate numerically, with a computational cost equivalent to the Black-Scholes formula. The smaller the parameters μ and σ are, the smaller the maturity T is, or the smaller the derivatives of the functions μ and σ with respect to their second variable are, the faster the convergence of the expansion is. This means that in practice, we need to calculate the expansion up to the second order, or possibly to the third order, to achieve an excellent accuracy (smaller than 2 bp on implied volatilities for various strikes and maturities). In addition, as a consequence of our approximation formulas, we establish that, for any fixed

[1] when explicitely stated, X may alternatively stand for the asset price.
[2] in some sense detailed later in this paper

maturity, a time dependent CEV model is equivalent to a CEV model with appropriate constant parameters (parameter averaging principle).

Comparison with the literature. In previous works, like Hagan et al in [HKLW02] for the SABR model, or Fouque et al in [FPS00] for stochastic volatility models, or Antonelli-Scarletti in [AS07], authors do a perturbation analysis with respect to a specific model parameter like the volatility, the mean reversion or the correlation. Their approach relies on a perturbation of the corresponding PDE. In contrast, we do not approximate the underlying PDE, or the related operator. We focus directly on the law of the random variable X_T^1 at maturity time, given its initial condition $X_0 = x_0$, using Malliavin calculus. Nicely, the extension to time dependent coefficients comes without any extra efforts.

Outline of the paper. In the following, we give some notations and assumptions used throughout the paper. The next section presents in an heuristic way our methodology to approximate the expected cost. We provide approximation formulas at the second and third order, using a log-normal or a normal proxy. In Section 3, we detail the approximation formulas for the case of time dependent CEV volatility. In Section 4, we analyse the magnitude of the correction and error terms of the general approximation formula (and at any order). The analysis depends on the payoff smoothness. The proofs of the main theorems 4.1-4.3-4.5 are postponed to section 5. In appendix 6, we bring together useful results to make our expansion explicit.

Definitions

Definition 1.1 As usual, we define $\mathscr{C}_0^\infty(\mathbb{R})$ as the space of real infinitely differentiable functions h with compact support. We also define $\mathscr{H}$ as the space of functions having at most an exponential growth. A function h belongs to $\mathscr{H}$ if $|h(x)| \leq c_1 e^{c_2|x|}$ for any x, for two constants c_1 and c_2.

Notations

The following notation will be used extensively throughout the paper.

Notation 1.1 ***Differentiation.***
If these derivatives have a meaning, we write:

- $\psi_t^{(i)}(x) = \frac{\partial^i \psi}{\partial x^i}(t,x)$ *for any function* ψ *of two variables.*
- $X_{i,t}^\varepsilon = \frac{\partial^i X_t^\varepsilon}{\partial \varepsilon^i}$ *is the* i^{th} *derivative of the parametrized process with respect to* ε.
- $X_{i,t} = \frac{\partial^i X_t^\varepsilon}{\partial \varepsilon^i}|_{\varepsilon=0}$ *. These processes play a crucial role in this work.*
- $\sigma_t = \sigma(t,x_0), \mu_t = \mu(t,x_0), \sigma_t^{(i)} = \sigma^{(i)}(t,x_0), \mu_t^{(i)} = \mu^{(i)}(t,x_0)$.

The following notation of Greeks will be useful for interpreting the expansion terms.

Notation 1.2 ***Greeks.***
Let Z be a random variable. Given a payoff function h, we define the i^{th} *Greek for the variable Z by the quantity (if it has a meaning) :*

$$\text{Greek}_i^h(Z) = \frac{\partial^i \mathbb{E}[h(Z+x)]}{\partial x^i}|_{x=0}.$$

Assumptions. In order to derive accurate approximations, we may assume that coefficients σ and μ are smooth enough. In what follows, N is an integer greater than 4.

- ***Assumption*** *(R_N). The functions σ and μ are bounded and of class C^N w.r.t x. Their derivatives up to order N are bounded.*

This assumption may be restrictive because σ and μ have to be bounded as well their derivatives. Actually, this statement is made only to simplify a bit our analysis, but we can prove that our approximation remains valid if some boundedness requirements are partially relaxed.

Notation 1.3 ***Function amplitudes.***
Under (R_N), we set

$$M_0 = max(|\sigma|_\infty, \cdots, |\sigma^{(N)}|_\infty, |\mu|_\infty, \cdots, |\mu^{(N)}|_\infty), \tag{1.4}$$
$$M_1 = max(|\sigma^{(1)}|_\infty, \cdots, |\sigma^{(N)}|_\infty, |\mu^{(1)}|_\infty, \cdots, |\mu^{(N)}|_\infty). \tag{1.5}$$

Although M_0 and M_1 may depend on N, we remove this dependence in our notation, for the sake of simplicity.

Remark 1.2 The constant M_0 measures the amplitude of the objective functions μ, σ and their derivatives w.r.t. the second variable, whereas M_1 measures only the amplitude of their derivatives. Notice that $M_1 \leq M_0$ and in case of deterministic functions σ and μ, one has $M_1 = 0$.

To perform the infinitesimal analysis, we rely on smoothness properties not related to the payoff function itself but rather to the law of the underlying stochastic models.

- ***Assumption*** *(E). The function σ does not vanish and its oscillation is bounded, meaning $1 \leq \frac{|\sigma|_\infty}{\sigma_{inf}} \leq C_E$ where $\sigma_{inf} = \inf_{(t,x)\in\mathbb{R}^+\times\mathbb{R}} \sigma(t,x)$.*

The assumption (E) is commonly called an ellipticity assumption.

We also need to divide our analysis according to the payoff smoothness. We split our analysis into three cases.

- ***Assumption*** *(H_1). h belongs to $\mathscr{C}_0^\infty(\mathbb{R})$. This case corresponds to smooth payoffs.*
- ***Assumption*** *(H_2). h and $h^{(1)}$ belongs to $\mathscr{H}$. This case corresponds to vanilla options (call-put).*
- ***Assumption*** *(H_3). h belongs to $\mathscr{H}$. This is the case of binary options (digital).*

2 Smart Taylor Development

In the following, we provide several approximation formulas, at the second and third order. These formulas are different if X models the logarithm of the underlying asset price or if it models directly the asset price. In the first case, our approximation is equivalent to take a lognormal proxy (or Black-Scholes proxy) for the asset price; in the second case, it is equivalent to use a normal proxy.

2.1 Second order approximation

Here, we consider that the dynamics (1.1) for X models the logarithm of the underlying asset. In the case of call option, the payoff is then $h(x) = (e^x - K)_+$, where K is the strike price.

Our perturbation approach relies on the Taylor expansion of the parameterized process. We have paved the way in our previous work [BGM08]. For the sake of completeness, we briefly recall the main steps to achieve a closed approximative formula.

From the definitions, $X_{i,t} \equiv \frac{\partial^i X_t^\varepsilon}{\partial \varepsilon^i}|_{\varepsilon=0}$, we can expand the perturbed process X_T^ε as follows:

$$X_T^\varepsilon = X_T^\varepsilon|_{\varepsilon=0} + \varepsilon X_{1,T} + \frac{\varepsilon^2}{2!} X_{2,T} + \ldots \tag{2.1}$$

Indeed, under the assumption (R_5), almost surely for any t, X_t^ε is C^4 w.r.t ε (see Theorem 2.3 in [Kun84]). The diffusion dynamics of $(X_{i,t}^\varepsilon \equiv \frac{\partial^i X_t^\varepsilon}{\partial \varepsilon^i})_{t\geq 0}$ is obtained by a straight differentiation of the parameters of the diffusion equation of X^ε. The first order term $X_{1,t}^\varepsilon$ is easily obtained as follows:

$$\begin{aligned} dX_{1,t}^\varepsilon =& \sigma_t(X_t^\varepsilon)dW_t + \mu_t(X_t^\varepsilon)dt \\ &+ \varepsilon X_{1,t}^\varepsilon(\sigma_t^{(1)}(X_t^\varepsilon)dW_t + \mu_t^{(1)}(X_t^\varepsilon)dt), X_{1,0}^\varepsilon = 0. \end{aligned} \tag{2.2}$$

From the definitions, we have $\sigma_t \equiv \sigma(t,x_0)$, $\mu_t \equiv \mu(t,x_0)$, $\sigma_t^{(i)} \equiv \sigma^{(i)}(t,x_0)$ and $\mu_t^{(i)} \equiv \mu^{(i)}(t,x_0)$. Then, we obtain

$$\begin{aligned} dX_{1,t} =& \sigma_t dW_t + \mu_t dt, X_{1,0} = 0, \\ dX_{2,t} =& 2X_{1,t}(\sigma_t^{(1)} dW_t + \mu_t^{(1)} dt), X_{2,0} = 0. \end{aligned}$$

Applying the expansion (2.1) at $\varepsilon = 1$, we conclude that $x_0 + X_{1,T}$ is a proxy for X_T. This is a Gaussian proxy for X, hence a lognormal proxy for the asset price (or Black-Scholes diffusion proxy). It justifies the notation

$$X_T^{BS} = x_0 + X_{1,T} = x_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s. \tag{2.3}$$

To obtain an approximation formula, we use the Taylor formula twice: first, for X_T^1 at the second order w.r.t ε around x_0, secondly for smooth function h at the first order w.r.t x around X_T^{BS}. This leads to:

$$\mathbb{E}[h(X_T)] = \mathbb{E}[h(X_T^{BS} + \frac{X_{2,T}}{2} + \ldots)] = \mathbb{E}[h(X_T^{BS})] + \mathbb{E}[h^{(1)}(X_T^{BS})\frac{X_{2,T}}{2}] + \ldots$$

To achieve an explicit formula, it remains to transform the correction term involving $X_{2,T}$ into a summation of greeks computed in the Black-Scholes proxy. This is performed using the Malliavin calculus. We refer to [BGM08] where the computations are detailed, or to the proof of Theorem 2.3 in this paper. This leads to the following theorem, which is a particular case of Theorem 2.1 of [BGM08] when there is no jump.

Theorem 2.1 ***(Second order approximation price formula using lognormal proxy).*** *Assume that the process (X_t) fulfills (R_5) and (E), and that the payoff function fulfills one of the assumptions (H_1), (H_2) or (H_3). Then*

$$\mathbb{E}[h(X_T)] = \mathbb{E}[h(X_T^{BS})] + \sum_{i=1}^{3} \alpha_{i,T} \mathrm{Greek}_i^h(X_T^{BS}) + \mathrm{Resid}_2, \tag{2.4}$$

where

$$\begin{aligned}
\alpha_{1,T} &= \int_0^T \mu_t (\int_t^T \mu_s^{(1)} ds) dt, \\
\alpha_{2,T} &= \int_0^T (\sigma_t^2 (\int_t^T \mu_s^{(1)} ds) + \mu_t (\int_t^T \sigma_s \sigma_s^{(1)} ds)) dt, \\
\alpha_{3,T} &= \int_0^T \sigma_t^2 (\int_t^T \sigma_s \sigma_s^{(1)} ds) dt.
\end{aligned}$$

Additionally, estimates of the error term Resid$_2$ *(otherwise stated as residual terms) are given in Theorems 4.1, 4.3 and 4.5, according to the cases (H_1), (H_2) or (H_3).*

Formula (2.4) is refereed as a second order approximation formula because we establish, in Theorem 4.3 for call/put option, that the error term Resid$_2$ is of order three with respect to the amplitudes of coefficients.
The above approximation of the price is a sum of two terms:

1. $\mathbb{E}[h(X_T^{BS})]$ is the leading order, corresponding to the price when the parameters σ and μ are deterministic. In the case of call/put option, it is given by the Black-Scholes formula. For other payoffs, we can use numerical integration because the density of the random variable X_T^{BS} is explicit.
2. $\sum_{i=1}^3 \alpha_{i,T} \mathrm{Greek}_i^h(X_T^{BS})$ are the volatility and drift correction terms, which depend on the first derivatives of μ and σ. These terms can be computed as easily as the main term.

The above formula may be simplified when the asset (i.e. $(e^{X_t})_{t\geq 0}$) is a martingale under the pricing measure[3] (also refered to Dupire model). Then, $\mu(t,x) = -\frac{1}{2}\sigma^2(t,x)$ and the formula writes

$$\mathbb{E}[h(X_T)] = \mathbb{E}[h(X_T^{BS})] + C_{1,T} (\frac{1}{2} \mathrm{Greek}_1^h(X_T^{BS}) - \frac{3}{2} \mathrm{Greek}_2^h(X_T^{BS}) + \mathrm{Greek}_3^h(X_T^{BS})) + \mathrm{Resid}_2$$

with

$$C_{1,T} = \int_0^T \sigma_t^2 (\int_t^T \sigma_s \sigma_s^{(1)} ds) dt. \tag{2.5}$$

[3] for instance, when one models the evolution of the forward price.

2.2 Third order approximation using a lognormal proxy

If the original model is close to its lognormal proxy, the formula (2.4) is very accurate (see the numerical results in Section 3). Otherwise, we can obtain higher accuracy by adding third order correction terms. The following result provides explicit expressions for these terms in the Dupire model ($\mu(t,x) = -\frac{1}{2}\sigma^2(t,x)$) for vanilla payoffs. Before, we introduce an appropriate definition, which will enable us to represent the coefficients of the greeks as iterated time integrals.

Definition 2.2 Integral Operator.
The integral operator ω^T is defined as follows: for any integrable function l, we set

$$\omega(l)_t^T = \int_t^T l_u du$$

for $t \in [0,T]$. Its n-times iteration is defined analogously: for any integrable functions $(l_1, \cdots, l_n)$, we set

$$\omega(l_1, \cdots, l_n)_t^T = \omega(l_1 \omega(l_2, \cdots, l_n)_\cdot^T)_t^T$$

for $t \in [0,T]$.

Theorem 2.3 ***(Third order approximation price formula in the Dupire model using lognormal proxy)****. Assume that the process $(X_t)_{t\geq 0}$ fulfills (R_7) and (E), and that the payoff function fulfills the assumption (H_2). Then*

$$\mathbb{E}[h(X_T)] = \mathbb{E}[h(X_T^{BS})] + \sum_{i=1}^{6} \eta_{i,T} \mathrm{Greek}_i^h(X_T^{BS}) + Resid_3, \tag{2.6}$$

where

$$\begin{aligned}
\eta_{1,T} =& \frac{C_{1,T}}{2} - \frac{C_{2,T}}{2} - \frac{C_{3,T}}{2} - \frac{C_{4,T}}{4} - \frac{C_{5,T}}{4} - \frac{C_{6,T}}{2},\\
\eta_{2,T} =& -\frac{3C_{1,T}}{2} + \frac{C_{2,T}}{2} + \frac{C_{3,T}}{2} + \frac{5C_{4,T}}{4} + \frac{5C_{5,T}}{4} + \frac{7C_{6,T}}{2} + \frac{C_{7,T}}{2} + \frac{C_{8,T}}{4},\\
\eta_{3,T} =& C_{1,T} - 2C_{4,T} - 2C_{5,T} - 6C_{6,T} - 3C_{7,T} - \frac{3C_{8,T}}{2},\\
\eta_{4,T} =& C_{4,T} + C_{5,T} + 3C_{6,T} + \frac{13C_{7,T}}{2} + \frac{13C_{8,T}}{4},\\
\eta_{5,T} =& -6C_{7,T} - 3C_{8,T},\\
\eta_{6,T} =& 2C_{7,T} + C_{8,T},
\end{aligned}$$

and

$$\begin{aligned}
C_{1,T} &= \omega(\sigma^2, \sigma\sigma^{(1)})_0^T, & C_{2,T} &= \omega(\sigma^2, (\sigma^{(1)})^2)_0^T,\\
C_{3,T} &= \omega(\sigma^2, \sigma\sigma^{(2)})_0^T, & C_{4,T} &= \omega(\sigma^2, \sigma^2, (\sigma^{(1)})^2)_0^T,\\
C_{5,T} &= \omega(\sigma^2, \sigma^2, \sigma\sigma^{(2)})_0^T, & C_{6,T} &= \omega(\sigma^2, \sigma\sigma^{(1)}, \sigma\sigma^{(1)})_0^T,\\
C_{7,T} &= \omega(\sigma^2, \sigma^2, \sigma\sigma^{(1)}, \sigma\sigma^{(1)})_0^T, & C_{8,T} &= \omega(\sigma^2, \sigma\sigma^{(1)}, \sigma^2, \sigma\sigma^{(1)})_0^T.
\end{aligned}$$

In addition, the estimate of the error term $Resid_3$ is given in Theorem 4.3.

An application of Theorem 4.3 yields that $Resid_3$ is of order four with respect to the volatility coefficient.

The proof of Theorem 2.3 is postponed to subsection 6.3.

2.3 Third order approximation using a normal proxy

In the previous third order approximation formula, numerous correction terms appear because the *smart expansion* involves simultaneously the volatility and the drift coefficients. If we consider directly a model on the asset price (and not on its logarithm), our expansion simplifies much because the drift in the Dupire model vanishes:

$$dX_t = \sigma(t, X_t)dW_t. \tag{2.7}$$

The above function σ for the asset price X and the volatility function σ in (1.1) for the log-asset are different, they are simply related by a change of variables of exponential type. Similarly, here the call payoff is equal to $h(x) = (x-K)_+$. Then, we can perform our expansion approach using the parametrized process X^ε that solves $dX_t^\varepsilon = \varepsilon\sigma(t, X_t^\varepsilon)dW_t$. We obtain that the model proxy for the asset price is defined by

$$X_t^N = x_0 + \int_0^t \sigma(s, x_0)dW_s, \tag{2.8}$$

which is a Gaussian process. We call it normal proxy. Formal computations of our *smart expansion* are analogous to those done for the lognormal proxy. We will skip details regarding the proof and the assumptions. We do not provide a rigorous estimation of the error term, we prefer to focus on the expressions of correction terms to achieve a third order approximation formula.

Theorem 2.4 ***(Third order approximation price formula in the Dupire model using normal proxy)****. For a vanilla payoff h, we have*

$$\mathbb{E}[h(X_T)] = \mathbb{E}[h(X_T^N)] + \sum_{i=1}^{6} \eta_{i,T}\mathrm{Greek}_i^h(X_T^N) + Error, \tag{2.9}$$

where

$$\eta_{1,T} = 0, \qquad \eta_{2,T} = -\frac{C_{2,T}}{2} + \frac{C_{3,T}}{2}, \qquad \eta_{3,T} = C_{1,T},$$
$$\eta_{4,T} = C_{4,T} + C_{5,T} + 3C_{6,T}, \qquad \eta_{5,T} = 0, \qquad \eta_{6,T} = 2C_{7,T} + C_{8,T}.$$

The coefficients $(C_{j,T})_{1\le j\le 8}$ are defined as in Theorem 2.3.

In the case of call/put option, the computations of the main term $\mathbb{E}[h(X_T^N)]$ and of the related greeks $(\mathrm{Greek}_i^h(X_T^N))_{1\le i\le 6}$ are straightforward because the proxy (2.8) is normal. Numerical results are reported in Section 3.

If one prefers to restrict to a second order approximation formula, it simpy writes

$$\mathbb{E}[h(X_T)] = \mathbb{E}[h(X_T^N)] + C_{1,T}\mathrm{Greek}_3^h(X_T^N) + Error. \tag{2.10}$$

2.4 Parameter averaging in CEV model

The time dependent CEV model on the underlying asset is defined by

$$dX_t = \nu_t X_t^{\beta_t} dW_t.$$

We suppose that the risk-free rate $(r_t)_t$ and the dividend yield $(q_t)_t$ are both deterministic. For simplicity in the following discussion, we assume $X_0 = 1$ in order to have a normalized model.

As discussed in [BGM08], the time dependent CEV model is interesting because it generates all the possible values of $(\sigma_t)_t$ and $(\sigma_t^{(1)})_t$ by appropriate choices of $(\nu_t)_t$ and $(\beta_t)_t$. Thus, in view of (2.4) and (2.10), this model may potentially generate all the possible prices at the second order.

When the coefficients $(\nu_t)_t$ and $(\beta_t)_t$ are constant, there is a closed formula for the call price (see [Sch89]). For general time dependent coefficients, we may use our approximation formulas based on log-normal or normal proxy. Alternatively, we may look for an equivalent CEV model with constant coefficients $\bar{\nu}$ and $\bar{\beta}$, with which the prices coincide at the second order. This is possible maturity by maturity. This principle has been studied for stochastic volatility models by Piterbarg [Pit05]. Owing to our approximation formulas, we retrieve that

$$\bar{\nu} = \sqrt{\frac{\int_0^T \nu_t^2 dt}{T}}, \qquad \bar{\beta} = \int_0^T \beta_t \rho_t dt, \qquad \text{with } \rho_t = \frac{\nu_t^2 \int_0^t \nu_s^2 ds}{\int_0^T \nu_t^2 \int_0^t \nu_s^2 ds}. \tag{2.11}$$

Proof In the context of lognormal proxy (β close to 1), we take

$$\sigma(t,x) = \nu_t e^{(\beta_t - 1)x}, \quad \mu(t,x) = -\frac{1}{2}\sigma^2(t,x), \quad h(x) = e^{-\int_0^T r_s ds}(e^{\int_0^T (r_s - q_s)ds} e^x - K)_+.$$

Then, our approximation formula (2.4) depends only on two constants $\int_0^T \nu_t^2 dt$ and $\int_0^T \nu_t^2 \int_t^T (\beta_s - 1)\nu_s^2 ds dt$. Consequently, two models must coincide with respect these two quantities in order to provide the same approximation formula (with lognormal proxy) up to second order. This easily leads to the identification (2.11).
When the model is close to normal proxy (β close to 0), we take

$$\sigma(t,x) = \nu_t x^{\beta_t}, \qquad \mu(t,x) = 0, \qquad h(x) = e^{-\int_0^T r_s ds}(e^{\int_0^T (r_s - q_s)ds} x - K)_+.$$

Then, using a similar approach based on formula (2.10), one retrieves exactly the same averaged parameters (2.11).

We conjecture that the averaging rule (2.11) is true not only for β close to 0 or 1, but also for various values in between. A numerical result (see Table 3.4) illustrates this averaging property. □

3 Numerical Experiments

In this section, we compare approximation formulas given in Theorem 2.1, Theorem 2.3 and Theorem 2.4, applied to Dupire model for call option. We assume that the risk-free rate and the dividend yield are both set at 0. For the following numerical results, we choose a CEV-type function for the local volatility. When the model is applied directly to the asset price (see (2.7) and Theorem 2.4), we have

$$\sigma(t,x) = \nu_t x^{\beta_t}, \qquad \mu(t,x) = 0, \qquad h(x) = (x-K)_+.$$

When the model is used for the log-asset price (see (1.1), Theorems 2.1 and 2.3), we have

$$\sigma(t,x) = \nu_t e^{(\beta_t - 1)x}, \qquad \mu(t,x) = -\frac{1}{2}\sigma^2(t,x), \qquad h(x) = (e^x - K)_+.$$

When the functions $(\nu_t)_t$ and $(\beta_t)_t$ do not depend on time (and thus are constant), we use the closed formula for call price [Sch89] as a benchmark. Otherwise, for time dependent functions, we use PDE methods to obtain reference values.

3.1 Accuracy of the second order formula (2.4) (based on a log-normal proxy)

*Constant parameters.*In the case of time independent volatility, the coefficient $C_{1,T}$ becomes:

$$C_{1,T} = \sigma_0^3 \sigma_0^{(1)} \frac{T^2}{2}.$$

In Table 3.1, we report related numerical results, which show that our formula is very accurate (errors in implied volatilities are smaller[4] than 2 bp) for β close to 1. This is coherent with the estimate of the error term $Resid_2$, because this model is close to the lognormal one. In Table 3.2, analogous tests are reported with $\beta = 0.2$. Here,

Table 3.1 Errors on implied Black-Scholes volatilities (in bp) between the second order approximation formula (2.4) and the closed formula for CEV model, expressed as a function of maturities in fractions of years and relative strikes. Parameters: $\beta = 0.8$, $\nu = 0.2$ and $x_0 = 0$.

T/K	80%	90%	100%	110%	120%
6M	-1.63	-0.22	-0.08	-0.17	-0.86
1Y	-1.11	-0.26	-0.15	-0.22	-0.63
1.5Y	-0.98	-0.32	-0.21	-0.28	-0.60
2Y	-0.95	-0.38	-0.28	-0.34	-0.62
3Y	-0.98	-0.51	-0.41	-0.46	-0.69
5Y	-1.16	-0.77	-0.67	-0.70	-0.89
10Y	-1.70	-1.37	-1.26	-1.27	-1.40

[4] 1 bp on implied volatilities is equal to 0.01%.

the errors are roughly equal to 20 bp, which is quite satisfactory. This case motivates the use of the third order approximation formula to obtain a better accuracy, this is discussed in the following subsection (see Table 3.6).

Table 3.2 Errors on implied Black-Scholes volatilities (in bp) between the second order approximation formula (2.4) and the closed formula for CEV model, expressed as a function of maturities in fractions of years and relative strikes. Parameters: $\beta = 0.2$, $\nu = 0.2$ and $x_0 = 0$.

T/K	80%	90%	100%	110%	120%
6M	-22.85	-3.33	-1.07	-2.61	-14.87
1Y	-16.60	-4.07	-2.14	-3.21	-10.20
1.5Y	-15.21	-5.11	-3.21	-4.03	-9.31
2Y	-15.13	-6.23	-4.27	-4.92	-9.29
3Y	-16.36	-8.53	-6.39	-6.74	-10.12
5Y	-20.47	-13.19	-10.60	-10.42	-12.74
10Y	-32.01	-24.45	-20.77	-19.45	-20.26

*Piecewise constant parameters.*Here, the functions ν and β are piecewise constant on each interval $[T_i, T_{i+1}[$ for each $i \le n$. Therefore, $C_{1,.}$ can be calculated recursively

$$C_{1,T_{i+1}} = C_{1,T_i} + (T_{i+1} - T_i)\sigma_{T_i}\sigma_{T_i}^{(1)} \sum_{j=1}^{i-1} \sigma_{T_j}^2 (T_{j+1} - T_j) + \frac{(T_{i+1} - T_i)^2}{2} \sigma_{T_i}^3 \sigma_{T_i}^{(1)},$$

with $C_{1,T_1} = \sigma_0^3 \sigma_0^{(1)} \frac{T_1{}^2}{2}$. In our tests, the piecewise constant functions ν and β are equal respectively on each interval of the form $[\frac{i}{20}, \frac{i+1}{20}[$ to $25\% - i \times 0.11\%$ and $100\% - i \times 0.75\%$. Results given in Table 3.3 show that our second order approximation formula is still very accurate for time dependent parameters ν and β. Using

Table 3.3 Errors on implied Black-Scholes volatilities (in bp) between the second order approximation formula (2.4) and the PDE method, expressed as a function of maturities in fractions of years and relative strikes. Parameters: time dependent ν and β, $x_0 = 0$.

T/K	80%	90%	100%	110%	120%
6M	-0.67	-0.09	0.03	-0.07	-0.35
1Y	-0.44	0.10	0.06	-0.09	-0.26
1.5Y	-0.38	-0.13	0.09	0.11	-0.25
2Y	0.37	0.15	-0.11	-0.14	-0.26

the same time dependent coefficients, we test the parameter averaging principle, that is described in paragraph 2.4. Results are reported in Table 3.4. The accuracy is still very good.

Table 3.4 Errors on implied Black-Scholes volatilities (in bp) between the closed CEV formula applied to an equivalent CEV model (2.11) and the PDE method, expressed as a function of relative strikes. Parameters: time dependent ν and β, $x_0 = 0$ and $T = 1Y$.

T/K	80%	90%	100%	110%	120%
1Y	0,09	-0,27	-0,20	-0,07	0,00

3.2 Accuracy of the third order formula (2.6)

Constant parameters. Tables 3.5 and 3.6 show that the third order approximation (2.6) is very good for various values of β. The use of this formula has much improved the accuracy in the case $\beta = 0.2$, for which the model is not close to the log-normal proxy.

Table 3.5 Error in implied Black-Scholes volatilities (in bp) between the third order approximation formula (2.6) and the closed formula for CEV model, expressed as a function of maturities in fractions of years and relative strikes. Parameters: $\beta = 0.8$, $\nu = 0.2$ and $x_0 = 0$.

T/K	80%	90%	100%	110%	120%
6M	-0.08	-0.02	-0.01	0.00	0.00
1Y	-0.06	-0.03	-0.01	-0.01	0.00
1.5Y	-0.06	-0.03	-0.02	-0.01	0.00
2Y	-0.06	-0.04	-0.02	-0.01	0.00
3Y	-0.08	-0.05	-0.03	-0.01	0.00
5Y	-0.10	-0.06	-0.04	-0.01	0.01
10Y	-0.16	-0.10	-0.06	-0.02	0.01

Table 3.6 Error in implied Black-Scholes volatilities (in bp) between the third order approximation formula (2.6) and the closed formula for CEV model, expressed as a function of maturities in fractions of years and relative strikes. Parameters: $\beta = 0.2$, $\nu = 0.2$ and $x_0 = 0$.

T/K	80%	90%	100%	110%	120%
6M	-1.23	-0.18	-0.01	0.12	0.53
1Y	-0.93	-0.34	-0.03	0.22	0.52
1.5Y	-1.19	-0.51	-0.06	0.31	0.68
2Y	-1.51	-0.68	-0.09	0.39	0.85
3Y	-2.22	-1.05	-0.19	0.52	1.17
5Y	-3.71	-1.87	-0.47	0.67	1.69
10Y	-7.32	-4.13	-1.56	0.55	2.38

3.3 Accuracy of the third order formula using normal approximation

Constant parameters. Tables 3.7 and 3.8 show that the third order approximation (2.9) is also very good for various values of β. The computation of this formula is slightly quicker than that with a log-normal proxy, because there are fewer terms.

Table 3.7 Error in implied Black-Scholes volatilities (in bp) between the third order approximation formula (2.9) and the closed formula for CEV model, expressed as a function of maturities in fractions of years and relative strikes. Parameters: $\beta = 0.8$, $\nu = 0.2$ and $x_0 = 0$.

T/K	80%	90%	100%	110%	120%
6M	-1.61	-0.07	-0.01	0.03	0.77
1Y	-0.88	-0.08	-0.02	0.03	0.45
1.5Y	-0.61	-0.11	-0.02	0.04	0.31
2Y	-0.51	-0.15	-0.03	0.06	0.25
3Y	-0.49	-0.23	-0.05	0.10	0.23
5Y	-0.71	-0.44	-0.11	0.16	0.30
10Y	-1.70	-1.09	-0.37	0.22	0.56

Table 3.8 Error in implied Black-Scholes volatilities (in bp) between the third order approximation formula (2.9) and the closed formula for CEV model, expressed as a function of maturities in fractions of years and relative strikes. Parameters: $\beta = 0.2$, $\nu = 0.2$ and $x_0 = 0$.

T/K	80%	90%	100%	110%	120%
6M	0.22	0.06	-0.01	-0.06	-0.16
1Y	0.41	0.11	0.00	-0.10	-0.26
1.5Y	0.56	0.17	0.00	-0.13	-0.34
2Y	0.71	0.24	0.02	-0.16	-0.41
3Y	1.02	0.39	0.06	-0.20	-0.53
5Y	1.75	0.79	0.21	-0.23	-0.71
10Y	4.71	2.55	1.15	0.10	-0.84

4 General results about error analysis

In this section, we analyse the error terms according to the payoff smoothness (smooth, vanilla or binary). To accomplish this, we first give some notations that will be used throughout the theorems and the proofs. Then, we provide a general expansion formula of the price $\mathbb{E}[h(X_T)]$ at any order, making explicit the order of magnitude of each term. This expansion is different according to the payoff smoothness: smooth payoff in Theorem 4.1, vanilla payoff in Theorem 4.3 under an additional ellipticity condition on σ and binay payoff in Theorem 4.5.

For the three cases, we discuss the form of error estimates. We show that the second order approximation formula (2.4) (and those at any order) is accurate under one of the following conditions:

- the maturity of the option T is small.
- the derivatives of the volatility σ and the drift μ w.r.t. the second variables are small. This is measured by the constant M_1 defined in (1.5). In particular, the model and the proxy coincide ($X \equiv X^{BS}$) when these derivatives vanish ($M_1 = 0$, see remark 1.2). This is coherent with our estimates since the correction and the error terms are estimated as $O(M_1)$ where O is the Landau symbol.
- The volatility, the drift and their derivatives are small. This dependence is represented using the constant M_0 defined in (1.4).

Moreover, when the three conditions are all satisfied, the approximation formula becomes even more accurate.

All the proofs are given in Section 5.

Notations.

- *About floating constants and upper bounds.* In the following statements and proofs, for the upper bounds we use numerous constants, that are not relabelled during the computations. We simply use the unique notation
$$A \le_c B$$
to assert that $A \le cB$, where c is a positive constant depending on the model parameters M_0, M_1, T, C_E (defined in assumption (E)) and on other universal constants. The constant c remains bounded when the model parameters go to 0, and it is uniform w.r.t. the parameter $\varepsilon \in [0,1]$. When informative, we make clear the dependence of upper bounds w.r.t. M_0, M_1 and T.
- *Model differentiation.* In the proofs, the derivatives of the parameterized process X^ε are useful: they are defined by $X^\varepsilon_{i,t} = \frac{\partial^i X^\varepsilon_t}{\partial \varepsilon^i}$ when these derivatives have a meaning. Additionally, we write:
$$Y^\varepsilon_T = X^\varepsilon_T - (x_0 + \varepsilon X_{1,T}), \qquad Y^\varepsilon_{k,i,T} = \frac{\partial^i((Y^\varepsilon_T)^k)}{\partial \varepsilon^i}, \qquad Y_{k,i,T} = Y^0_{k,i,T},$$
$$R_{k,i,T} = \frac{\int_0^1 Y^{(1-\lambda)}_{k,i+1,T} \lambda^i d\lambda}{i!}.$$
- *Miscellaneous.* As usual, the $\mathbf{L}_p$-norm of a real random variable Z is denoted by $\|Z\|_p = [\mathbb{E}|Z|^p]^{1/p}$.

4.1 Error analysis for smooth payoff

Theorem 4.1 ***Asymptotic expansion for the price of smooth payoff (*** $h \in \mathscr{C}_0^\infty(R)$***).*** *For $m \ge 2$ assume that (R_{m+2}) holds. If the payoff h fulfills Assumption (H_1), then one has*
$$\mathbb{E}[h(X_T)] = \mathbb{E}[h(X_T^{BS})] + \sum_{i=2}^{m} Ord_i + Resid_m, \tag{4.1}$$

where different terms are estimated as follows.

- *The contribution for order* $i \in \{2,\dots,m\}$: $Ord_i = \sum_{k=1}^{\lfloor \frac{i}{2} \rfloor} \mathbb{E}[h^{(k)}(X_T^{BS}) \frac{Y_{k,i,T}}{k!i!}]$ *and it is estimated by*

$$|Ord_i| \le_c \sup_{1\le j\le \lfloor \frac{i}{2} \rfloor -1} |h^{(j)}|_\infty M_1 M_0^{i-1} (\sqrt{T})^i. \qquad (4.2)$$

- *The residual term for order m is :* $Resid_m = \mathbb{E}[\sum_{k=1}^{[\frac{m}{2}]} h^{(k)}(X_T^{BS}) \frac{R_{k,m,T}}{k!} + \frac{(Y_T^1)^{\lfloor \frac{m}{2} \rfloor+1}}{\lfloor \frac{m}{2} \rfloor!} \int_0^1 h^{(\lfloor \frac{m}{2} \rfloor+1)}(vX_T + (1-v)(X_T^{BS}))(1-v)^{\lfloor \frac{m}{2} \rfloor} dv]$, *such that*

$$|Resid_m| \le_c \sup_{1\le j\le \lfloor \frac{m}{2} \rfloor} |h^{(j)}|_\infty M_1 M_0^{m} (\sqrt{T})^{m+1}. \qquad (4.3)$$

In the *multiplicative case* ($\sigma(t,x) = \Delta a(t,x)$ and $\mu(t,x) = \Delta b(t,x)$), we have $M_0 \le_c \Delta$ and $M_1 \le_c \Delta$. Thus, we obtain

$$Ord_i = O((\Delta\sqrt{T})^i) \quad \text{for } 2 \le i \le m, \qquad Resid_m = O((\Delta\sqrt{T})^{m+1}).$$

This justifies that Equation (4.1) should be viewed as an approximation formula of order m.

Notice that the above theorem provides which terms have to be computed to achieve a given accuracy. But to effectively compute these terms as a summation of Greeks (as in Theorems 2.1 and 2.3), we shall use results in Appendix 6.

4.2 Error analysis for vanilla payoff

The payoff h for this kind of option is not necessarily smooth, it is almost everywhere differentiable and belongs to the space $\mathscr{H}$. The previous expansion in the case of smooth payoff is no more valid. Indeed, the i-th order contribution Ord_i has been represented using the derivatives of $h^{(1)}$ that do not necessarily exist anymore. Therefore we introduce some new variables in order to represent higher contributions only using $h^{(1)}$ (and not higher order derivatives).

Lemma 4.2 *Given* $m \ge 2$, *assume* (R_{3m-2}) *and* (E). *Let* $v \in [0,1]$. *There exist random variables* $(G_i)_{2\le i\le m}, S_m, I_{m,v} \in \cap_{p\ge 1} \mathbf{L}_p$ *such that for any* $l \in \mathscr{C}_0^\infty(\mathbb{R})$, *one has*

$$\sum_{k=1}^{i-1} \frac{1}{k!} \mathbb{E}[l^{(k)}(X_T^{BS}) \frac{Y_{k,k+i-1,T}}{(k+i-1)!}] = \mathbb{E}[l^{(1)}(X_T^{BS}) G_i] \quad \textit{for } 2 \le i \le m,$$

$$\sum_{k=1}^{m-1} \frac{1}{k!} \mathbb{E}[l^{(k)}(X_T^{BS}) R_{k,k+m-1,T}] = \mathbb{E}[l^{(1)}(X_T^{BS}) S_m],$$

$$\mathbb{E}[\frac{(Y_T^1)^m}{(m-1)!} l^{(m)}(vX_T + (1-v)X_T^{BS})] = \mathbb{E}[l^{(1)}(vX_T + (1-v)X_T^{BS}) I_{m,v}].$$

Additionally, we have for any $p \geq 1$

$$\|G_i\|_p \leq_c (\frac{M_0}{\sigma_{inf}})^{i-2} M_1 M_0^{i-1} (\sqrt{T})^i, \tag{4.4}$$

$$\|S_m\|_p + \sup_{v\in[0,1]} \|I_{m,v}\|_p \leq_c (\frac{M_0}{\sigma_{inf}})^{m-1} M_1 M_0^m (\sqrt{T})^{m+1}. \tag{4.5}$$

The proof of this lemma is postponed to Subsection 5.2.

The random variables in the above lemma are now used to represent conveniently successive contributions in the general approximation formula for vanilla payoffs. This is the following statement.

Theorem 4.3 ***Asymptotic expansion for the price of vanilla payoff (*** $h \in \mathscr{H}$ ***and*** $h' \in \mathscr{H}$***).***
Given $m \geq 2$, assume (R_{3m-2}) and (E). If the payoff h fulfills Assumption (H_2), then we have

$$\mathbb{E}[h(X_T)] = \mathbb{E}[h(X_T^{BS})] + \sum_{i=2}^{m} Ord_i + Resid_m, \tag{4.6}$$

where different terms are estimated as follows.

- *The contribution of order $i \in \{2,\dots,m\}$ is $Ord_i = \mathbb{E}[h^{(1)}(X_T^{BS})G_i]$ and it is estimated by:*

$$|Ord_i| \leq_c \|h^{(1)}(X_T^{BS})\|_2 (\frac{M_0}{\sigma_{inf}})^{i-2} M_1 M_0^{i-1} (\sqrt{T})^i. \tag{4.7}$$

- *The residual for order m is $Resid_m = \mathbb{E}[h^{(1)}(X_T^{BS})S_m] + \int_0^1 \mathbb{E}[h^{(1)}(vX_T + (1-v)X_T^{BS})I_{m,v}](1-v)^{m-1}dv$, such that*

$$|Resid_m| \leq_c (\|h^{(1)}(X_T^{BS})\|_2 + \sup_{v\in[0,1]} \|h^{(1)}(vX_T + (1-v)X_T^{BS})\|_2) (\frac{M_0}{\sigma_{inf}})^{m-1} M_1 M_0^m (\sqrt{T})^{m+1}. \tag{4.8}$$

Notice that the error term in Theorem 2.1 for vanilla payoff is $Resid_2$. For the third order approximation formula of Theorem 2.3, it is $Resid_3$. Let us comment on the above theorem.

- The label Ord_i is due to the fact that this term is bounded by $M_1 M_0^{i-1}(\sqrt{T})^i$ multiplied by an ellipticity factor of the form $(\frac{M_0}{\sigma_{inf}})^n$. This ellipticity factor is new compared to the case of smooth payoffs. To have a clear view on each contribution, one should have in mind the multiplicative case ($\sigma(t,x) = \Delta a(t,x)$ and $\mu(t,x) = \Delta b(t,x)$) which leads to $\max(M_1, M_0) \leq_c \Delta$ and

$$Ord_i = O((\Delta\sqrt{T})^i) \quad \text{for } 2 \leq i \leq m, \qquad Resid_m = O((\Delta\sqrt{T})^{m+1}).$$

  That is why we refer to Equation (4.6) as an approximation formula of order m.

- Correction terms are brought together in a different way than in the case of smooth payoffs. Indeed, the hierarchy (in terms of amplitudes) is modified according to the payoff smoothness. However, it is easy to check that the second order approximation is the same for smooth payoffs and vanilla ones. For higher orders, there is no more coincidence with the smooth case.
- Similarly to the smooth case, the above formula provides the appropriate terms to compute to reach a given level of accuracy. It remains to explicitely compute these terms as a summation of Greeks, using results in Appendix 6. This is done in Theorems 2.1 and 2.3 for $m=2$ and $m=3$.
- Finally to accommodate irregular payoffs, we require extra smoothness properties on μ and σ.

4.3 Error analysis for binary payoff

For this kind of option, the payoff h is not necessarily smooth, it is at least in $\mathscr{H}$. The results below are easy extensions of the case of vanilla options, we leave the proof to the reader.

Lemma 4.4 *Given $m \geq 1$, assume (R_{3m+2}) and (E). Let $v \in [0,1]$. There exist random variables $(P_i)_{1\leq i\leq m}, Q_m, T_{m,v} \in \cap_{p\geq 1}\mathbf{L}_p$ such that, for any $l \in \mathscr{C}_0^\infty(\mathbb{R})$, one has:*

$$\sum_{k=1}^{i} \frac{1}{k!}\mathbb{E}[l^{(k)}(X_T^{BS})\frac{Y_{k,k+i,T}}{(k+i)!}] = \mathbb{E}[l(X_T^{BS})P_i] \quad \textit{for } 1 \leq i \leq m,$$
$$\sum_{k=1}^{m} \frac{1}{k!}\mathbb{E}[l^{(k)}(X_T^{BS})R_{k,k+m,T}] = \mathbb{E}[l(X_T^{BS})Q_m],$$
$$\mathbb{E}[\frac{(Y_T^1)^{m+1}}{m!} l^{(m+1)}(vX_T+(1-v)X_T^{BS})] = \mathbb{E}[l(vX_T+(1-v)X_T^{BS})T_{m,v}].$$

Moreover, they are estimated in the $\mathbf{L}_p$ norm as follows:

$$\|P_i\|_p \leq_c (\frac{M_0}{\sigma_{inf}})^i M_1 M_0^{i-1}(\sqrt{T})^i,$$
$$\|Q_m\|_p + \sup_{v\in[0,1]} \|T_{m,v}\|_p \leq_c (\frac{M_0}{\sigma_{inf}})^{m+1} M_1 M_0^m (\sqrt{T})^{m+1}.$$

We are now in a position to state an expansion formula of order m.

Theorem 4.5 ***Asymptotic expansion for the price of binary payoff** ($h \in \mathscr{H}$)**.***
Given $m \geq 1$, assume (R_{3m+2}) and (E). If the payoff h fulfills Assumption (H_3), then we have

$$\mathbb{E}[h(X_T)] = \mathbb{E}[h(X_T^{BS})] + \sum_{i=1}^{m} Ord_i + Resid_m, \tag{4.9}$$

where different terms are as follows.

– *The contribution for order $i \in \{1,\dots,m\}$ is $Ord_i = \mathbb{E}[h(X_T^{BS})P_i]$ and it is estimated by:*

$$|Ord_i| \leq_c \|h(X_T^{BS})\|_2 (\frac{M_0}{\sigma_{inf}})^i M_1 M_0^{i-1} (\sqrt{T})^i. \tag{4.10}$$

– *The residual term for order m is $Resid_m = \mathbb{E}[h(X_T^{BS})Q_m] + \int_0^1 \mathbb{E}[h(vX_T + (1-v)X_T^{BS})T_{m,v}](1-v)^m dv$, such that*

$$|Resid_m| \leq_c (\|h(X_T^{BS})\|_2 + \sup_{v\in[0,1]} \|h(vX_T + (1-v)X_T^{BS})\|_2)$$
$$(\frac{M_0}{\sigma_{inf}})^{m+1} M_1 M_0^m (\sqrt{T})^{m+1}. \tag{4.11}$$

Notice that the second order approximation for smooth payoffs and vanilla options is only a first order approximation for binary options. This is due to the lack of regularity of the binary payoffs.

5 Proofs

For the following, we use the same definitions and notations as in chapter 1 of [Nua06]. Before giving the proofs for the main theorems, we need to upper bound the $\mathbf{L}_p$ norm of the derivatives $X_{i,t}^{\varepsilon}$ to state Theorem 4.1, to upper bound also the $\mathbf{L}_p$ norm of the Malliavin derivatives $D^j_{t_1,\cdots,t_j} X_{i,t}^{\varepsilon}$, and use the key lemma 5.5 in order to state Theorems 4.3 and 4.5.

5.1 Proof of Theorem 4.1 (Smooth payoff)

The proof of Theorem 4.1 is performed through two steps:

- **Step 1:** Upper bound the $\mathbf{L}_p$ norm of $X_{i,t}^{\varepsilon}$.
- **Step 2:** Completion of the proof of Theorem 4.1.

We first recall that $\varepsilon \to X_t^{\varepsilon}$ is almost surely C^{N-1} w.r.t. ε under assumption (R_N).

5.1.1 Step 1: Upper bounds for the $\mathbf{L}_p$ norm of $X_{i,t}^{\varepsilon}$

We aim at proving the following result, which may be useful, independently of our work.

Theorem 5.1 *Given $N \geq 2$, assume (R_N). For every $\varepsilon \in [0,1]$ and $p \geq 1$, we have*

$$\sup_{t\leq T} \|X_{1,t}^{\varepsilon}\|_p \leq_c M_0\sqrt{T}; \tag{5.1}$$
$$\sup_{t\leq T} \|X_{i,t}^{\varepsilon}\|_p \leq_c M_1 M_0^{i-1} (\sqrt{T})^i, \quad \forall i \in \{2,\dots,N-1\}. \tag{5.2}$$

A meaning of the first inequality is that the first derivative has the same amplitude as the implicit total standard deviation $M_0\sqrt{T}$. The second inequality shows that the bounds of the derivative estimates decrease successively by the implicit total standard deviation $M_0\sqrt{T}$. Furthermore, the dependence w.r.t. the constant M_1 shows that the derivatives $(X_{i,t})_{i\geq 2}$ are null if the function σ and μ are deterministic (see Remark 1.2). In this case, X is the Black-Scholes model.

Proof The existence of any moment is easy to establish, we will skip details. In the following, we rather focus on their dependence w.r.t. M_0, M_1 and $\sqrt{T}$.

Clearly, it is sufficient to prove estimates for $p \geq 2$. Take $p \geq 2$, note that $X^{\varepsilon}_{1,.}$ is the solution of the linear SDE:

$$\begin{aligned} dX^{\varepsilon}_{1,t} &= \sigma_t(X^{\varepsilon}_t)dW_t + \mu_t(X^{\varepsilon}_t)dt + \varepsilon X^{\varepsilon}_{1,t}(\sigma^{(1)}_t(X^{\varepsilon}_t)dW_t + \mu^{(1)}_t(X^{\varepsilon}_t)dt), \\ X^{\varepsilon}_{1,0} &= 0. \end{aligned}$$

To estimate the $\mathbf{L}_p$ norm of the solution of the above *linear* equation, we state a lemma, that will be repeatedly used in the following computations.

Lemma 5.2 *Assume that Z is an Itô process such that*

i) $\sup_{t\leq T}\|Z_t\|_p < +\infty$ *for some* $p \geq 2$*;*
ii) Z solves a linear equation

$$Z_t = \int_0^t Z_s(a_s dW_s + b_s ds) + \int_0^t \alpha_s dW_s + \beta_s ds,$$

where $\sup_{t\leq T}(\|\alpha_t\|_p + \|\beta_t\|_p) < +\infty$*, a and b are bounded.*

Then, for a constant c (depending only on p and T), we have

$$\sup_{t\leq T}\|Z_t\|_p \leq c \sup_{t\leq T}(\|\alpha_t\|_p + \|\beta_t\|_p)\sqrt{T}e^{c(|a|_\infty + |b|_\infty)^p T^{P/2}}. \tag{5.3}$$

The proof is quite standard: it results from easy calculations using BDG inequalities and Gronwall's lemma. We omit further details.

From this, it readily follows that

$$\sup_{t\leq T}\|X^{\varepsilon}_{1,t}\|_p \leq_c \max(|\sigma|_\infty, |\mu|_\infty)\sqrt{T} \leq_c M_0\sqrt{T}.$$

This proves the first inequality (5.1).

We now prove the second inequality (5.2) which is not straightforward. To accomplish this, it is useful to scale the parameters. Let us define the new variables:

$$\tilde{X}^{\varepsilon}_t = X_t^{\frac{\varepsilon}{M_0\sqrt{T}}}, \tag{5.4}$$

$$\tilde{\sigma}(t,x) = \frac{\sigma(t,x)}{M_0}, \qquad \tilde{\mu}(t,x) = \frac{\mu(t,x)}{M_0}. \tag{5.5}$$

From Equation (1.3), one obtains the dynamics of the rescaled process $(\tilde{X}^{\varepsilon}_t)_t$:

$$d\tilde{X}^{\varepsilon}_t = \varepsilon(\tilde{\sigma}_t(\tilde{X}^{\varepsilon}_t)\frac{dW_t}{\sqrt{T}} + \tilde{\mu}_t(\tilde{X}^{\varepsilon}_t)\frac{dt}{\sqrt{T}}), \tilde{X}^{\varepsilon}_0 = x_0, \tag{5.6}$$

where $\varepsilon \in [0, M_0\sqrt{T}]$. The advantage of this change of parameters is that the constant M_0 associated to the new coefficients $\tilde{\sigma}$ and $\tilde{\mu}$ is bounded by 1 (thus, it is model-free):

$$\max(|\tilde{\sigma}|_\infty, \cdots, |\tilde{\sigma}^{(N)}|_\infty, |\tilde{\mu}|_\infty, \cdots, |\tilde{\mu}^{(N)}|_\infty) = 1.$$

Additionally, there is a simple relation between derivatives of X^ε and those of $\tilde{X}^\varepsilon$:

$$\tilde{X}^\varepsilon_{i,t} \equiv \frac{\partial^i(\tilde{X}^\varepsilon_t)}{\partial \varepsilon^i} = \frac{\partial^i(X_t^{\frac{\varepsilon}{M_0\sqrt{T}}})}{\partial \varepsilon^i} = \frac{1}{(M_0\sqrt{T})^i} X_{i,t}^{\frac{\varepsilon}{M_0\sqrt{T}}}.$$

Using this notation, the proof of Inequality (5.2) is reduced to prove that

$$\sup_{t \le T} \|\tilde{X}^\varepsilon_{i,t}\|_p \le_c \frac{M_1}{M_0}. \tag{5.7}$$

for every $\varepsilon \in [0, M_0\sqrt{T}]$ and $i \in \{2, \ldots, N-1\}$.

Proof of (5.7) . By successive differentiation of (5.6), it is not hard to prove

$$\sup_{t \le T} \|\tilde{X}^\varepsilon_{i,t}\|_p \le_c 1. \tag{5.8}$$

Indeed, we obtain linear SDEs[5] solved by $\tilde{X}^\varepsilon_{i,.}$, to which we can apply Lemma 5.2. It gives uniform bounds because the arising processes (a, b, α, β) are proportional to $1/\sqrt{T}$ and then multiplied by $\sqrt{T}$ in Lemma 5.2. Another heuristic argument, to get that the bound (5.8) is indeed equal to 1, is the following: on the one hand, the integrands $\frac{W_t}{\sqrt{T}}$ and $\frac{t}{\sqrt{T}}$ in the SDE (5.6) are $O(1)$ over the maturity T. On the other hand, the uniform bounds for the derivatives of $\tilde{\sigma}$ and $\tilde{\mu}$ up to order N are smaller than 1. Consequently, the $\mathbf{L}_p$ estimates (5.8) remain uniformly bounded.

However, the inequality (5.8) is not equivalent to the inequality (5.7) because $\frac{M_1}{M_0} \le 1$. But this preliminary estimate is useful to establish the final one as follows. To prove the required inequality, we first show that $\tilde{X}^\varepsilon_{i,.}$ solves a linear equation, this is stated in the following proposition.

Proposition 5.3 *Given $N \ge 2$, assume (R_N). For $2 \le i \le N-1$, $\tilde{X}_{i,.}$ is the solution of the linear SDE:*

$$\begin{aligned} d\tilde{X}^\varepsilon_{i,t} &= dH^\varepsilon_{i,t} + \tilde{X}^\varepsilon_{i,t} dL^\varepsilon_t, \qquad \tilde{X}^\varepsilon_{i,0} = 0, \\ dL^\varepsilon_t &= \varepsilon(\tilde{\sigma}^{(1)}_t(\tilde{X}_t)\frac{dW_t}{\sqrt{T}} + \tilde{\mu}^{(1)}_t(\tilde{X}_t)\frac{dt}{\sqrt{T}}), \\ dH^\varepsilon_{i,t} &= P^\varepsilon_{\tilde{\sigma},i,t}\frac{dW_t}{\sqrt{T}} + P^\varepsilon_{\tilde{\mu},i,t}\frac{dt}{\sqrt{T}}, \end{aligned} \tag{5.9}$$

where the processes $(P^\varepsilon_{\tilde{\sigma},i,t})_{t\ge 0}$ and $(P^\varepsilon_{\tilde{\mu},i,t})_{t\ge 0}$ are defined in the proof.

---

[5] this is fully justified in Proposition 5.3.

Proof Take $i \geq 2$, the SDE for the i^{th} derivative is obtained from Equation (5.6) using differentiation under the integral sign (see [Kun84]):

$$d\tilde{X}^{\varepsilon}_{i,t} = \frac{\partial^i(\varepsilon\tilde{\sigma}_t(\tilde{X}^{\varepsilon}_t))}{\partial\varepsilon^i}\frac{dW_t}{\sqrt{T}} + \frac{\partial^i(\varepsilon\tilde{\mu}_t(\tilde{X}^{\varepsilon}_t))}{\partial\varepsilon^i}\frac{dt}{\sqrt{T}}, \qquad \tilde{X}^{\varepsilon}_{i,0} = 0. \tag{5.10}$$

The application of the Leibniz formula for the i^{th} derivative of the product (that is $(\varepsilon f(\varepsilon))^{(i)} = \varepsilon f^{(i)}(\varepsilon) + i f^{(i-1)}(\varepsilon)$) gives:

$$\frac{\partial^i(\varepsilon\tilde{\sigma}_t(\tilde{X}^{\varepsilon}_t))}{\partial\varepsilon^i} = \varepsilon\frac{\partial^i(\tilde{\sigma}_t(\tilde{X}^{\varepsilon}_t))}{\partial\varepsilon^i} + i\frac{\partial^{i-1}(\tilde{\sigma}_t(\tilde{X}^{\varepsilon}_t))}{\partial\varepsilon^{i-1}}.$$

Using the Faà di Bruno formula for derivative of composite function (apply Lemma 6.4 with $g(x) = \tilde{\sigma}_t(x)$ and $f(\varepsilon) = \tilde{X}^{\varepsilon}_t$), one obtains

$$\begin{aligned}\frac{\partial^i(\varepsilon\tilde{\sigma}(t,\tilde{X}^{\varepsilon}_t))}{\partial\varepsilon^i} =& \varepsilon \sum_{\substack{k=(k_1,\cdots,k_i)\in\mathbb{N}^i \\ \sum_{j=1}^{i} jk_j = i}} d_k \tilde{\sigma}_t^{(\sum_{j=1}^{i} k_j)}(\tilde{X}^{\varepsilon}_t)\prod_{j=1}^{i}(\tilde{X}^{\varepsilon}_{j,t})^{k_j} \\ &+ i \sum_{\substack{k=(k_1,\cdots,k_{i-1})\in\mathbb{N}^{i-1} \\ \sum_{j=1}^{i-1} jk_j = i-1}} d_k \tilde{\sigma}_t^{(\sum_{j=1}^{i-1} k_j)}(\tilde{X}^{\varepsilon}_t)\prod_{j=1}^{i-1}(\tilde{X}^{\varepsilon}_{j,t})^{k_j}.\end{aligned}$$

Notice that the i^{th} component k_i can take only two values 0 or 1 (because $ik_i \leq \sum_{j=1}^{i} jk_j = i$). When $k_i = 1$, one has $k_j = 0$ for $j < i$ and $d_k = 1$ (see Lemma 6.4). Thus, we obtain

$$\begin{aligned}\frac{\partial^i(\varepsilon\tilde{\sigma}(t,\tilde{X}^{\varepsilon}_t))}{\partial\varepsilon^i} =& \varepsilon\tilde{\sigma}_t^{(1)}(\tilde{X}^{\varepsilon}_t)\tilde{X}^{\varepsilon}_{i,t} \\ &+ \varepsilon \sum_{\substack{k=(k_1,\cdots,k_{i-1},0)\in\mathbb{N}^i \\ \sum_{j=1}^{i-1} jk_j = i}} d_k \tilde{\sigma}_t^{(\sum_{j=1}^{i-1} k_j)}(\tilde{X}^{\varepsilon}_t)\prod_{j=1}^{i-1}(\tilde{X}^{\varepsilon}_{j,t})^{k_j} \\ &+ i \sum_{\substack{k=(k_1,\cdots,k_{i-1})\in\mathbb{N}^{i-1} \\ \sum_{j=1}^{i-1} jk_j = i-1}} d_k \tilde{\sigma}_t^{(\sum_{j=1}^{i-1} k_j)}(\tilde{X}^{\varepsilon}_t)\prod_{j=1}^{i-1}(\tilde{X}^{\varepsilon}_{j,t})^{k_j} \\ :=& \varepsilon\tilde{\sigma}_t^{(1)}(\tilde{X}^{\varepsilon}_t)\tilde{X}^{\varepsilon}_{i,t} + P^{\varepsilon}_{\tilde{\sigma},i,t}.\end{aligned} \tag{5.11}$$

We define analogously $P^{\varepsilon}_{\tilde{\mu},i,t}$ by replacing $\tilde{\sigma}$ by $\tilde{\mu}$ in the expression (5.11). It writes

$$\frac{\partial^i(\varepsilon\tilde{\mu}(t,\tilde{X}^{\varepsilon}_t))}{\partial\varepsilon^i} = \varepsilon\tilde{\mu}_t^{(1)}(\tilde{X}^{\varepsilon}_t)\tilde{X}^{\varepsilon}_{i,t} + P^{\varepsilon}_{\tilde{\mu},i,t}. \tag{5.12}$$

The two equalities (5.11) and (5.12) plugged into the relation (5.10) give immediately the result. □

End of proof of (5.7). Owing to Equation (5.3), $\tilde{X}_{i,.}$ is the solution of a linear SDE, to which we apply Lemma 5.2. We obtain

$$\sup_{t\le T}\|\tilde{X}^{\varepsilon}_{i,t}\|_p \le_c \sup_{t\le T}\|P^{\varepsilon}_{\tilde{\sigma},i,t}\|_p + \sup_{t\le T}\|P^{\varepsilon}_{\tilde{\mu},i,t}\|_p.$$

In view of the expression of $P^{\varepsilon}_{\tilde{\sigma},i,t}$ in Equation (5.11), using the Hölder inequality and the preliminary estimates (5.8), we obtain

$$\sup_{t\le T}\|P^{\varepsilon}_{\tilde{\sigma},i,t}\|_p \le_c \sum_{\substack{k=(k_1,\cdots,k_{i-1},0)\in\mathbb{N}^i\\ \sum_{j=1}^{i-1} jk_j=i}} |\tilde{\sigma}^{(\sum_{j=1}^{i-1}k_j)}|_\infty + \sum_{\substack{k=(k_1,\cdots,k_{i-1})\in\mathbb{N}^{i-1}\\ \sum_{j=1}^{i-1} jk_j=i-1}} |\tilde{\sigma}^{(\sum_{j=1}^{i-1}k_j)}|_\infty$$

Since $\sum_{j=1}^{i-1} jk_j \ge 1$ and k_j are integers, we have $\sum_{j=1}^{i-1} k_j \ge 1$. It readily follows

$$\begin{aligned}\sup_{t\le T}\|P^{\varepsilon}_{\tilde{\sigma},i,t}\|_p &\le_c \max(|\tilde{\sigma}^{(1)}|_\infty,\cdots,|\tilde{\sigma}^{(N-1)}|_\infty) = c\frac{\max(|\sigma^{(1)}|_\infty,\cdots,|\sigma^{(N-1)}|_\infty)}{M_0}\\ &\le_c \frac{M_1}{M_0}.\end{aligned}$$

The same inequality holds for $P^{\varepsilon}_{\tilde{\mu},i,t}$, which finishes the proof of (5.7). Consequently, Theorem 5.1 is proved. □

5.1.2 Step 2: Proof of Theorem 4.1 (Smooth payoff)

Before performing the Taylor expansion, we recall the notations:

$$\begin{aligned}&Y^{\varepsilon}_T = X^{\varepsilon}_T - (x_0+\varepsilon X_{1,T}), \qquad Y^{\varepsilon}_{k,i,T} = \frac{\partial^i((Y^{\varepsilon}_T)^k)}{\partial\varepsilon^i}, \qquad Y_{k,i,T} = Y^0_{k,i,T},\\ &R_{k,i,T} = \frac{\int_0^1 Y^{(1-\lambda)}_{k,i+1,T}\lambda^i d\lambda}{i!}.\end{aligned}$$

Clearly one has $X_T = X_T^{BS} + Y_T^1$. We write

$$\begin{aligned}
\mathbb{E}[h(X_T)] =&\mathbb{E}[h(X_T^{BS})] + \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor} \frac{1}{k!}\mathbb{E}[h^{(k)}(X_T^{BS})(Y_T^1)^k] \\
&+ \int_0^1 \mathbb{E}[\frac{(Y_T^1)^{\lfloor \frac{m}{2} \rfloor+1}(1-v)^{\lfloor \frac{m}{2} \rfloor}}{\lfloor \frac{m}{2} \rfloor !} h^{(\lfloor \frac{m}{2} \rfloor+1)}(vX_T + (1-v)X_T^{BS})]dv \\
=&\mathbb{E}[h(X_T^{BS})] + \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor} \frac{1}{k!}\mathbb{E}[h^{(k)}(X_T^{BS})(\sum_{i=2k}^{m} \frac{Y_{k,i,T}}{i!} + R_{k,m,T})] \\
&+ \int_0^1 \mathbb{E}[\frac{(Y_T^1)^{\lfloor \frac{m}{2} \rfloor+1}(1-v)^{\lfloor \frac{m}{2} \rfloor}}{\lfloor \frac{m}{2} \rfloor !} h^{(\lfloor \frac{m}{2} \rfloor+1)}(vX_T + (1-v)X_T^{BS})]dv \\
=&\mathbb{E}[h(X_T^{BS})] + \sum_{i=2}^{m} \sum_{k=1}^{[\frac{i}{2}]} \frac{1}{k!}\mathbb{E}[h^{(k)}(X_T^{BS})\frac{Y_{k,i,T}}{i!}] \\
&+ \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor} \frac{1}{k!}\mathbb{E}[h^{(k)}(X_T^{BS})R_{k,m,T}] \\
&+ \int_0^1 \mathbb{E}[\frac{(Y_T^1)^{\lfloor \frac{m}{2} \rfloor+1}(1-v)^{\lfloor \frac{m}{2} \rfloor}}{\lfloor \frac{m}{2} \rfloor !} h^{(\lfloor \frac{m}{2} \rfloor+1)}(vX_T + (1-v)X_T^{BS})]dv \\
=&\mathbb{E}[h(X_T^{BS})] + \sum_{i=2}^{m} Ord_i + Resid_m,
\end{aligned}$$

where we have used a Taylor expansion twice for the two first identities (notice that $Y_{k,i,T} = 0$ for $i \leq 2k-1$), and we have interchanged the summations for the third one. The equation (4.1) is proved.

Now we establish estimates (4.2) and (4.3). By differentiation of composite function using the Faà di Bruno formula (see Lemma 6.4 with $g(x) = x^k$ and $f(\varepsilon) = Y_T^\varepsilon$), we obtain

$$Y_{k,i,T}^{\varepsilon} = \sum_{\substack{\alpha=(\alpha_1,\cdots,\alpha_i)\in\mathbb{N}^i \\ \sum_{j=1}^i j\alpha_j = i,\ \sum_{j=1}^i \alpha_j \leq k}} d_\alpha \frac{k!}{(\sum_{j=1}^i \alpha_j)!} (Y_T^\varepsilon)^{k-\sum_{j=1}^i \alpha_j} \prod_{j=1}^{i} (Y_{1,j,T}^\varepsilon)^{\alpha_j}. \tag{5.13}$$

Here, we restrict to the indices α such that $\sum_{j=1}^i \alpha_j \leq k$ because we have $g^{(\sum_{j=1}^i \alpha_j)}(x) = 0$ when $\sum_{j=1}^i \alpha_j > k$. Using Equation (5.2), one deduces for each $j \in \{2,\ldots,i\}$ that

$$\|Y_{1,j,T}^\varepsilon\|_p = \|X_{j,T}^\varepsilon\|_p \leq_c M_1 M_0^{j-1} (\sqrt{T})^j \tag{5.14}$$

for any $p \geq 1$. For $j = 1$, the inequality (5.14) is also available because we can write $Y_{1,1,T}^\varepsilon = \int_0^\varepsilon X_{2,T}^\lambda d\lambda$, which readily implies

$$\|Y_{1,1,T}^\varepsilon\|_p \leq_c M_1 M_0 (\sqrt{T})^2 \leq_c M_1 \sqrt{T}.$$

For any indices α, we have the rough estimate $\|(Y_T^\varepsilon)^{k-\sum_{j=1}^i \alpha_j}\|_p \le_c 1$. Using the above estimate, (5.14) and the Hölder inequality, we finally get

$$\begin{aligned}
\|Y^\varepsilon_{k,i,T}\|_p &\le_c \sum_{\substack{\alpha=(\alpha_1,\cdots,\alpha_i)\in\mathbb{N}^i\\ \sum_{j=1}^i j\alpha_j=i}} \prod_{j=1}^i (M_1 M_0^{j-1}(\sqrt{T})^j)^{\alpha_j}\\
&\le_c \sum_{\substack{\alpha=(\alpha_1,\cdots,\alpha_i)\in\mathbb{N}^i\\ \sum_{j=1}^i j\alpha_j=i}} (\frac{M_1}{M_0})^{\sum_{j=1}^i \alpha_j} M_0^{\sum_{j=1}^i j\alpha_j}(\sqrt{T})^{\sum_{j=1}^i j\alpha_j}\\
&\le_c \frac{M_1}{M_0} M_0^i (\sqrt{T})^i = cM_1 M_0^{i-1}(\sqrt{T})^i, \qquad (5.15)
\end{aligned}$$

where we used $\frac{M_1}{M_0} \le 1$ and $\sum_{j=1}^i \alpha_j \ge 1$ (since $(\alpha_j)_j$ are integers that satisfy $\sum_{j=1}^i j\alpha_j = i \ge 1$). The inequality (5.15) gives immediately the inequality (4.2). It also leads to

$$\|R_{k,m,T}\|_p \le_c M_1 M_0^m (\sqrt{T})^{m+1}. \qquad (5.16)$$

Since $Y_T^1 = X_{2,T} + R_{1,2,T}$, one has

$$\|Y_T^1\|_p \le \|X_{2,T}\|_p + \|R_{1,2,T}\|_p \le_c M_1 M_0 (\sqrt{T})^2 + M_1 M_0^2 (\sqrt{T})^3 \le_c M_1 M_0 (\sqrt{T})^2$$

(recall our definition of generic constants). Therefore

$$\|(Y_T^1)^{\lfloor \frac{m}{2}\rfloor+1}\|_p \le_c M_1^{\lfloor \frac{m}{2}\rfloor+1} M_0^{\lfloor \frac{m}{2}\rfloor+1} (\sqrt{T})^{2(\lfloor \frac{m}{2}\rfloor+1)} \le_c M_1 M_0^m (\sqrt{T})^{m+1}, \qquad (5.17)$$

where we have used $M_1 \le M_0$ and $2\lfloor \frac{m}{2}\rfloor \ge m-1$. The inequalities (5.16) and (5.17) readily leads to the inequality (4.3). The proof is complete. □

5.2 Proof of Lemma 4.2

For Malliavin calculus, we use the notation of Nualart [Nua06] for the Sobolev spaces $\mathbb{D}_{k,p}$ associated to the norm $\|.\|_{k,p}$. We divide the proof of Lemma 4.2 into three steps:

- **Step 1**: Upper bounds for the $\mathbb{D}^{k,p}$ norm of $X^\varepsilon_{i,t}$.
- **Step 2**: Statement of a suitable integration by parts formula (Lemma 5.5) in order to handle the irregularity of vanilla payoffs.
- **Step 3**: Completion of the proof of Lemma 4.2.

In all this subsection, we assume (R_{3m-2}) for a given $m \ge 2$.

5.2.1 Step 1: Upper Bounds for the $\mathbb{D}^{k,p}$ norm of $X^\varepsilon_{i,t}$

The aim of this paragraph is to show that, for every $\varepsilon \in [0,1]$, we have

- $X_T^\varepsilon \in \mathbb{D}^{3m-2,\infty}$ with

$$\|DX_T^\varepsilon\|_{3m-3,p} \le_c |\sigma|_\infty \sqrt{T}, \qquad (5.18)$$

- for each $i \in \{1,\dots,3m-3\}$, $X^{\varepsilon}_{i,T}$ belongs to $\mathbb{D}^{3m-3-i,\infty}$ with

$$\|X^{\varepsilon}_{1,T}\|_{3m-4,p} \le_c M_0\sqrt{T}, \tag{5.19}$$

$$\|X^{\varepsilon}_{i,T}\|_{3m-3-i,p} \le_c M_1 M_0^{i-1}(\sqrt{T})^i, \quad i \ge 2. \tag{5.20}$$

Only the proofs of upper bounds need few details. To prove the inequality (5.18), we use the following lemma.

Lemma 5.4 *For any $t \in [0,T]$ and $\varepsilon \in [0,1]$, X^{ε}_t belongs to $\mathbb{D}^{3m-2,\infty}$. Moreover, the j first Malliavin derivatives of X^{ε}_t satisfy the following estimates:*

$$\sup_{(t_1,\cdots,t_j)\in[0,T]^j,t\in[0,T]} \|D^j_{t_1,\cdots,t_j}X^{\varepsilon}_t\|_p \le_c |\sigma|_\infty.$$

Proof We first take $j=1$; for $t_1 \in [0,t]$, using formula (2.59) in [Nua06] p.126, we have

$$D_{t_1}X^{\varepsilon}_t = \varepsilon\sigma(t_1, X^{\varepsilon}_{t_1})e^{\int_{t_1}^t \varepsilon(\sigma^{(1)}_s(X^{\varepsilon}_s)dW_s + (\mu^{(1)}_s - \varepsilon\frac{(\sigma^{(1)}_s)^2}{2})(X^{\varepsilon}_s)ds)}.$$

This leads to the announced estimate when $j=1$. The result for $j \ge 2$ is easily obtained by induction. □

From the definition of the $\mathbb{D}^{k,p}$ norm, it follows that

$$\begin{aligned}\|DX^{\varepsilon}_T\|_{3m-3,p} &= (\sum_{j=1}^{3m-3}\mathbb{E}[(\int_0^T\cdots\int_0^T (D_{t_1,\cdots,t_j}X^{\varepsilon}_T)^2 dt_1\cdots dt_j)^{\frac{p}{2}}])^{\frac{1}{p}} \\ &\le (\sum_{j=1}^{3m-3} T^{\frac{j}{2}} \sup_{(t_1,\cdots,t_j)\in[0,T]^j} \|D^j_{t_1,\cdots,t_j}X^{\varepsilon}_T\|^p_p)^{\frac{1}{p}} \le_c |\sigma|_\infty\sqrt{T}\end{aligned}$$

using Lemma 5.4 at the last inequality. This proves the first inequality (5.18).
Now, to establish the upper bounds (5.19) and (5.20), we note that it is equivalent to prove, for every $\varepsilon \in [0, M_0\sqrt{T}]$, that

$$\|\tilde{X}^{\varepsilon}_{1,T}\|_{3m-4,p} \le_c 1, \tag{5.21}$$

$$\|\tilde{X}^{\varepsilon}_{i,T}\|_{3m-3-i,p} \le_c \frac{M_1}{M_0}, \quad i \ge 2, \tag{5.22}$$

where $(\tilde{X}^{\varepsilon}_t)_t$ is the rescaled process introduced in (5.4). Using similar arguments as for (5.8), we obtain

$$\|\tilde{X}^{\varepsilon}_{i,T}\|_{N-1-i,p} \le_c 1, \tag{5.23}$$

for any $\varepsilon \in [0, M_0\sqrt{T}]$. The inequality (5.21) is proved but not (5.22), because $\frac{M_1}{M_0} \le 1$. To establish (5.22), we proceed as for the proof of Theorem 5.1. We will skip further details.

5.2.2 Step 2: Statement of the integration by part Lemma

To handle non-smooth payoffs, our computations rely on a non-degenerate condition on the volatility (stated in assumption (E)). This type of condition is essential to prove the following lemma.

Lemma 5.5 *Assume (E) and (R_{k+1}) for a given $k \geq 1$. Let Z belong to $\cap_{p\geq 1}\mathbb{D}^{k,p}$. For any $v \in [0,1]$, there exists a random variable Z_k^v in any $\mathbf{L}_p$ ($p \geq 1$) such that for any function $l \in \mathscr{C}_0^\infty(\mathbb{R})$, we have*

$$\mathbb{E}[l^{(k)}(vX_T + (1-v)X_T^{BS})Z] = \mathbb{E}[l(vX_T + (1-v)X_T^{BS})Z_k^v].$$

Moreover, we have $\|Z_k^v\|_p \leq_c \frac{\|Z\|_{k,2p}}{(\sigma_{inf}\sqrt{T})^k}$, uniformly in v.

This is a straightforward adaptation of Lemma 5.3 in [BGM08], we omit the proof.

5.2.3 Step 3: Proof of Lemma 4.2

Starting from Equation (5.13) with $i+k-1$ instead of i, we write

$$Y^\varepsilon_{k,i+k-1,T} = \sum_{\substack{\alpha=(\alpha_1,\cdots,\alpha_i)\in\mathbb{N}^i \\ \sum_{j=1}^{i+k-1} j\alpha_j = i+k-1, \sum_{j=1}^{i+k-1}\alpha_j \leq k}} d_\alpha \frac{k!}{(\sum_{j=1}^{i+k-1}\alpha_j)!}(Y_T^\varepsilon)^{k-\sum_{j=1}^{i+k-1}\alpha_j}\prod_{j=1}^{i+k-1}(Y^\varepsilon_{1,j,T})^{\alpha_j}.$$

Using Equation (5.20) one deduces, for $2 \leq j \leq i+k-1$, that

$$\|Y^\varepsilon_{1,j,T}\|_{k-1,p} = \|X^\varepsilon_{j,T}\|_{k-1,p} \leq_c M_1 M_0^{j-1}(\sqrt{T})^j.$$

This inequality is also available for $j = 1$, since

$$\|Y^\varepsilon_{1,1,T}\|_{k-1,p} = \|\int_0^\varepsilon X^\lambda_{2,T}d\lambda\|_{k-1,p} \leq_c M_1 M_0(\sqrt{T})^2 \leq_c M_1\sqrt{T}.$$

Additionally, we note that $Y^\varepsilon_{k,i+k-1,T} \in \mathbb{D}^{k-1,\infty}$. Furthermore, using the Hölder inequality for the spaces $\mathbb{D}^{k-1,\infty}$ (see Proposition 1.5.6 in [Nua06]), we obtain

$$\|Y^\varepsilon_{k,i+k-1,T}\|_{k-1,p} \leq_c M_1 M_0^{i+k-2}(\sqrt{T})^{i+k-1}. \tag{5.24}$$

We omit the details of the above computations because they are very similar to those used for (5.15). Then, Lemma 5.5 ensures the existence of a random variable G_i in $\mathbf{L}_p$. Its $\mathbf{L}_p$ norm is estimated using Lemma 5.5 and Inequality (5.24):

$$\|G_i\|_p \leq_c \sum_{k=1}^{i-1}\frac{M_1 M_0^{i+k-2}(\sqrt{T})^{i+k-1}}{(\sigma_{inf}\sqrt{T})^{k-1}} \leq_c \frac{M_1 M_0^{2(i-1)-1}(\sqrt{T})^i}{\sigma_{inf}^{i-2}},$$

using $\frac{M_0}{\sigma_{inf}} \geq 1$. For S_m and $I_{m,v}$, we proceed analogously.

5.3 Statement of Theorem 4.3 (Vanilla options)

We first assume that h is a smooth function. We have

$$\begin{aligned}
\mathbb{E}[h(X_T)] =& \mathbb{E}[h(X_T^{BS})] \\
&+ \sum_{k=1}^{m-1} \frac{1}{k!} \mathbb{E}[h^{(k)}(X_T^{BS})(\sum_{i=k+1}^{m} \frac{Y_{k,k+i-1,T}}{(k+i-1)!} + R_{k,k+m-1,T})] \\
&+ \int_0^1 \mathbb{E}[\frac{(Y_T^1)^m(1-v)^{m-1}}{(m-1)!} h^{(m)}(vX_T + (1-v)X_T^{BS})]dv \\
=& \mathbb{E}[h(X_T^{BS})] + \sum_{i=2}^{m} \frac{1}{k!} \sum_{k=1}^{i-1} \mathbb{E}[h^{(k)}(X_T^{BS}) \frac{Y_{k,k+i-1,T}}{(k+i-1)!}] \\
&+ \sum_{k=1}^{m-1} \frac{1}{k!} \mathbb{E}[h^{(k)}(X_T^{BS}) R_{k,k+m-1,T}] \\
&+ \int_0^1 \mathbb{E}[\frac{(Y_T^1)^m(1-v)^{m-1}}{(m-1)!} h^{(m)}(vX_T + (1-v)X_T^{BS})]dv \\
=& \mathbb{E}[h(X_T^{BS})] + \sum_{i=2}^{m} \mathbb{E}[h^{(1)}(X_T^{BS}) G_i] \\
&+ \mathbb{E}[h^{(1)}(X_T^{BS}) S_m] \\
&+ \int_0^1 \mathbb{E}[h^{(1)}(vX_T + (1-v)X_T^{BS}) I_{m,v}](1-v)^{m-1} dv,
\end{aligned}$$

where we have used a Taylor expansion in the first identity, interchanged the summations in the second equality, and used the Lemma 4.2 in the last one. So yields the identity (4.6) for smooth payoff.
Additionally, using estimates (4.4) and (4.5) from Lemma 4.2, it is straightforward to deduce the inequalities (4.7) and (4.8).
It remains to extend the result to vanilla options (instead of smooth function h). Since all the estimates depend only on $h^{(1)}$, it can be achieved by a standard density argument. We refer to [BGM08] for details.

6 Appendix

Here, we bring together the results (and their proofs) which allow us to derive the explicit terms in the formulas (2.4), (2.6), (2.9) and (2.10).
In the following, (u_t) (resp. (v_t)) is a square integrable and predictable (resp. deterministic) process and l is a smooth function with compact support.

6.1 Technical results related to explicit correction terms

The two first lemmas are proved in [BGM08].

Lemma 6.1 *For any continuous (or piecewise continuous) function f, any continuous semimartingale Z vanishing at t=0, one has:*

$$\int_0^T f_t Z_t dt = \int_0^T (\int_t^T f_s ds) dZ_t.$$

Lemma 6.2 *One has:*

$$\mathbb{E}[(\int_0^T u_t dW_t) l(\int_0^T v_t dW_t)] = \mathbb{E}[(\int_0^T v_t u_t dt) l^{(1)}(\int_0^T v_t dW_t)].$$

In the case of deterministic u, it is equal to $\int_0^T v_t u_t dt\,\mathrm{Greek}_1^l(\int_0^T v_t dW_t)$.

6.2 Explicit correction in the case of Dupire model

In this case ($\mu \equiv -\frac{1}{2}\sigma^2$), the SDEs solved by the derivatives $X_{i,\cdot}$ become:

$$\begin{aligned}
dX_{1,t} =& \sigma_t dW_t - \frac{\sigma_t^2}{2} dt, X_{1,0} = 0,\\
dX_{2,t} =& 2X_{1,t}(\sigma_t^{(1)} dW_t - \sigma_t \sigma_t^{(1)} dt), X_{2,0} = 0,\\
dX_{3,t} =& 3(X_{2,t}(\sigma_t^{(1)} dW_t - \sigma\sigma_t^{(1)} dt) + (X_{1,t})^2(\sigma_t^{(2)} dW_t - (\sigma_t\sigma_t^{(2)} + (\sigma_t^{(1)})^2) dt)), X_{3,0} = 0.
\end{aligned}$$

Lemma 6.3 *We have*

$$\mathbb{E}[(\int_0^T v_t X_{1,t} dt) l(\int_0^T \sigma_t dW_t)] = \omega(\sigma^2, v)_0^T \big(\mathbb{E}[l^{(1)}(\int_0^T \sigma_t dW_t)] - \frac{1}{2}\mathbb{E}[l(\int_0^T \sigma_t dW_t)]\big), \tag{6.1}$$

$$\begin{aligned}
\mathbb{E}[(\int_0^T v_t \frac{X_{2,t}}{2} dt) l(\int_0^T \sigma_t dW_t)] = \omega(\sigma^2, \sigma\sigma^{(1)}, v)_0^T \big(&\mathbb{E}[l^{(2)}(\int_0^T \sigma_t dW_t)]\\
&- \frac{3}{2}\mathbb{E}[l^{(1)}(\int_0^T \sigma_t dW_t)] + \frac{1}{2}\mathbb{E}[l(\int_0^T \sigma_t dW_t)]\big),
\end{aligned} \tag{6.2}$$

$$\begin{aligned}
\mathbb{E}[(\int_0^T v_t \frac{(X_{1,t})^2}{2} dt) l(\int_0^T \sigma_t dW_t)] =& \omega(\sigma^2, \sigma^2, v)_0^T \big(\mathbb{E}[l^{(2)}(\int_0^T \sigma_t dW_t)] - \mathbb{E}[l^{(1)}(\int_0^T \sigma_t dW_t)]\big)\\
&+ (\frac{1}{4}\omega(\sigma^2, \sigma^2, v)_0^T + \frac{1}{2}\omega(\sigma^2, v)_0^T)\mathbb{E}[l(\int_0^T \sigma_t dW_t)],
\end{aligned} \tag{6.3}$$

$$\begin{aligned}
\mathbb{E}[(\int_0^T \sigma_t \sigma_t^{(1)} \frac{X_{2,t} X_{1,t}}{2} dt) l(\int_0^T \sigma_t dW_t)] =& (-\frac{3}{2}C_{6,T} - \frac{1}{2}C_{7,T} - \frac{1}{4}C_{8,T})\mathbb{E}[l(\int_0^T \sigma_t dW_t)]\\
&+ (2C_{6,T} + \frac{5}{2}C_{7,T} + \frac{5}{4}C_{8,T})\mathbb{E}[l^{(1)}(\int_0^T \sigma_t dW_t)]\\
&+ (-4C_{7,T} - 2C_{8,T})\mathbb{E}[l^{(2)}(\int_0^T \sigma_t dW_t)]\\
&+ (2C_{7,T} + C_{8,T})\mathbb{E}[l^{(3)}(\int_0^T \sigma_t dW_t)],
\end{aligned} \tag{6.4}$$

where

$$C_{6,T} = \omega(\sigma^2, \sigma\sigma^{(1)}, \sigma\sigma^{(1)})_0^T, \qquad C_{7,T} = \omega(\sigma^2, \sigma^2, \sigma\sigma^{(1)}, \sigma\sigma^{(1)})_0^T,$$
$$C_{8,T} = \omega(\sigma^2, \sigma\sigma^{(1)}, \sigma^2, \sigma\sigma^{(1)})_0^T.$$

Proof Applying first Lemma 6.1 to $f(t) = v_t$ and $Z_t = X_{1,t}$, we obtain:

$$\begin{aligned}\mathbb{E}[(\int_0^T v_t X_{1,t} dt) l(\int_0^T \sigma_t dW_t)] =& \mathbb{E}[(\int_0^T (\int_t^T v_s ds) dX_{1,t}) l(\int_0^T \sigma_t dW_t)]\\ =& \mathbb{E}[(\int_0^T (\int_t^T v_s ds)(\sigma_t dW_t - \frac{\sigma_t^2}{2} dt) l(\int_0^T \sigma_t dW_t)]\\ =& (\int_0^T \sigma_t^2 (\int_t^T v_s ds) dt) \mathbb{E}[l^{(1)}(\int_0^T \sigma_t dW_t)]\\ &- (\int_0^T \frac{\sigma_t^2}{2} (\int_t^T v_s ds) dt) \mathbb{E}[l(\int_0^T \sigma_t dW_t)],\end{aligned}$$

and we have used Lemma 6.2 for the last equality. This gives (6.1).

To establish the equalities (6.2), (6.3) and (6.4), we proceed analogously. We only detail the computations for (6.4). Using Lemma 6.1 ($f(t) = \sigma_t \sigma_t^{(1)}$, $Z_t = \frac{X_{2,t} X_{1,t}}{2}$) to justify the first following identity and Lemma 6.2 for the second one, we can write

$$\begin{aligned}&\mathbb{E}[(\int_0^T \sigma_t \sigma_t^{(1)} \frac{X_{2,t} X_{1,t}}{2} dt) l(\int_0^T \sigma_t dW_t)]\\ =& \mathbb{E}[(\int_0^T \omega(\sigma\sigma^{(1)})_t^T (X_{1,t}^2 (\sigma_t^{(1)} dW_t - \sigma_t \sigma_t^{(1)} dt)\\ &+ \frac{X_{2,t}}{2} (\sigma_t dW_t - \frac{\sigma_t^2}{2} dt) + \sigma_t \sigma_t^{(1)} X_{1,t} dt)) l(\int_0^T \sigma_t dW_t)]\\ =& \mathbb{E}[(\int_0^T \omega(\sigma\sigma^{(1)})_t^T (-\sigma_t \sigma_t^{(1)} X_{1,t}^2 - \sigma_t^2 \frac{X_{2,t}}{4} + \sigma_t \sigma_t^{(1)} X_{1,t}) dt) l(\int_0^T \sigma_t dW_t)]\\ &+ \mathbb{E}[(\int_0^T \omega(\sigma\sigma^{(1)})_t^T (\sigma_t \sigma_t^{(1)} X_{1,t}^2 + \sigma_t^2 \frac{X_{2,t}}{2}) dt) l^{(1)}(\int_0^T \sigma_t dW_t)].\end{aligned}$$

Then, we obtain the announced identity by an application of the three first identities (6.1), (6.2) and (6.3). □

6.3 Proof of Theorem 2.3

Proof Using Theorem 4.3 and Lemma 4.2, the price is approximated at the third order by

$$E[h(X_T^{BS})] + E[h^{(1)}(X_T^{BS}) \frac{X_{2,T}}{2}] + \mathbb{E}[h^{(1)}(X_T^{BS}) \frac{X_{3,T}}{3!}] + \mathbb{E}[h^{(2)}(X_T^{BS}) \frac{(\frac{X_{2,T}}{2})^2}{2}].$$

We compute each correction term separately.

Step 1: term with $X_{2,T}$. Owing to Lemma 6.2, we have

$$\begin{aligned}\mathbb{E}[h^{(1)}(X_T^{BS}) \frac{X_{2,T}}{2}] =& \mathbb{E}[h^{(1)}(X_T^{BS})(\int_0^T X_{1,t} (\sigma_t^{(1)} dW_t - \sigma_t \sigma_t^{(1)} dt))]\\ =& \mathbb{E}[h^{(2)}(X_T^{BS})(\int_0^T \sigma_t \sigma_t^{(1)} X_{1,t} dt)]\\ &- \mathbb{E}[h^{(1)}(X_T^{BS})(\int_0^T \sigma_t \sigma_t^{(1)} X_{1,t} dt)].\end{aligned}$$

Apply Lemma 6.3 (equality (6.1)) to obtain

$$\mathbb{E}[h^{(1)}(X_T^{BS})\frac{X_{2,T}}{2}] = C_{1,T}(\mathbb{E}[h^{(3)}(X_T^{BS})] - \frac{3}{2}\mathbb{E}[h^{(2)}(X_T^{BS})] + \frac{1}{2}\mathbb{E}[h^{(1)}(X_T^{BS})]),$$

where $C_{1,T} = \omega(\sigma^2, \sigma\sigma^{(1)})_0^T$.

Step 2: term with $X_{3,T}$. From Lemma 6.1 and 6.2, we obtain

$$\begin{aligned}\mathbb{E}[h^{(1)}(X_T^{BS})\frac{X_{3,T}}{3!}] =& \mathbb{E}[h^{(2)}(X_T^{BS})\frac{\int_0^T \sigma_t\sigma_t^{(1)}X_{2,t}dt}{2}] \\ &- \mathbb{E}[h^{(1)}(X_T^{BS})\frac{\int_0^T \sigma_t\sigma_t^{(1)}X_{2,t}dt}{2}] \\ &+ \mathbb{E}[h^{(2)}(X_T^{BS})\frac{\int_0^T \sigma_t\sigma_t^{(2)}(X_{1,t})^2dt}{2}] \\ &- \mathbb{E}[h^{(1)}(X_T^{BS})\frac{\int_0^T (\sigma_t\sigma_t^{(2)} + (\sigma_t^{(1)})^2)(X_{1,t})^2dt}{2}].\end{aligned}$$

An application of Lemma 6.3 (equalities (6.2) and (6.3)) gives:

$$\begin{aligned}\mathbb{E}[h^{(1)}(X_T^{BS})\frac{X_{3,T}}{3!}] =& (-\frac{1}{2}C_{2,T} - \frac{1}{2}C_{3,T} - \frac{1}{4}C_{4,T} - \frac{1}{4}C_{5,T} - \frac{1}{2}C_{6,T})\mathbb{E}[h^{(1)}(X_T^{BS})] \\ &+ (\frac{1}{2}C_{3,T} + C_{4,T} + \frac{5}{4}C_{5,T} + 2C_{6,T})\mathbb{E}[h^{(2)}(X_T^{BS})] \\ &+ (-C_{4,T} - 2C_{5,T} - \frac{5}{2}C_{6,T})\mathbb{E}[h^{(3)}(X_T^{BS})] \\ &+ (C_{5,T} + C_{6,T})\mathbb{E}[h^{(4)}(X_T^{BS})],\end{aligned}$$

where

$C_{2,T} = \omega(\sigma^2, (\sigma^{(1)})^2)_0^T$, $C_{3,T} = \omega(\sigma^2, \sigma\sigma^{(2)})_0^T$, $C_{4,T} = \omega(\sigma^2, \sigma^2, (\sigma^{(1)})^2)_0^T$,
$C_{5,T} = \omega(\sigma^2, \sigma^2, \sigma\sigma^{(2)})_0^T$, $C_{6,T} = \omega(\sigma^2, \sigma\sigma^{(1)}, \sigma\sigma^{(1)})_0^T$.

Step 3: term with $(X_{2,T})^2$. Similarly, we have

$$\begin{aligned}\mathbb{E}[h^{(2)}(X_T^{BS})\frac{(\frac{X_{2,T}}{2})^2}{2}] =& \mathbb{E}[h^{(2)}(X_T^{BS})\int_0^T (\sigma_t^{(1)}\frac{X_{1,t}X_{2,t}}{2}dW_t - \sigma_t\sigma_t^{(1)}\frac{X_{1,t}X_{2,t}}{2}dt + (\sigma_t^{(1)})^2\frac{X_{1,t}^2}{2}dt)] \\ =& \mathbb{E}[h^{(3)}(X_T^{BS})\int_0^T (\sigma_t\sigma_t^{(1)}\frac{X_{1,t}X_{2,t}}{2}dt] \\ &- \mathbb{E}[h^{(2)}(X_T^{BS})\int_0^T (\sigma_t\sigma_t^{(1)}\frac{X_{1,t}X_{2,t}}{2}dt] \\ &+ \mathbb{E}[h^{(2)}(X_T^{BS})\int_0^T (\sigma_t^{(1)})^2\frac{X_{1,t}^2}{2}dt].\end{aligned}$$

Using Lemma 6.3 (third and fourth equalities), it follows

$$\begin{aligned}\mathbb{E}[h^{(2)}(X_T^{BS})\frac{(\frac{X_{2,T}}{2})^2}{2}] =& (\frac{1}{2}C_{2,T}+\frac{1}{4}C_{4,T}+\frac{3}{2}C_{6,T}+\frac{1}{2}C_{7,T}+\frac{1}{4}C_{8,T})\mathbb{E}[h^{(2)}(X_T^{BS})]\\&+(-C_{4,T}-\frac{7}{2}C_{6,T}-3C_{7,T}-\frac{3}{2}C_{8,T})\mathbb{E}[h^{(3)}(X_T^{BS})]\\&+(C_{4,T}+2C_{6,T}+\frac{13}{2}C_{7,T}+\frac{13}{4}C_{8,T})\mathbb{E}[h^{(4)}(X_T^{BS})]\\&+(-6C_{7,T}-3C_{8,T})\mathbb{E}[h^{(5)}(X_T^{BS})]\\&+(2C_{7,T}+C_{8,T})\mathbb{E}[h^{(6)}(X_T^{BS})],\end{aligned}$$

where

$$\begin{aligned}C_{1,T}&=\omega(\sigma^2,\sigma\sigma^{(1)})_0^T, & C_{2,T}&=\omega(\sigma^2,(\sigma^{(1)})^2)_0^T,\\C_{4,T}&=\omega(\sigma^2,\sigma^2,(\sigma^{(1)})^2)_0^T, & C_{6,T}&=\omega(\sigma^2,\sigma\sigma^{(1)},\sigma\sigma^{(1)})_0^T,\\C_{7,T}&=\omega(\sigma^2,\sigma^2,\sigma\sigma^{(1)},\sigma\sigma^{(1)})_0^T, & C_{8,T}&=\omega(\sigma^2,\sigma\sigma^{(1)},\sigma^2,\sigma\sigma^{(1)})_0^T.\end{aligned}$$

Final step. To get the announced formula, we bring together all the previous contributions and use that $\mathbb{E}(h^{(i)}(X_T^{BS}))=\mathrm{Greek}_i^h(X_T^{BS})$. □

6.4 Faà di Bruno's formula [dB57]

Lemma 6.4 *If g and f are functions that are sufficiently differentiable, then*

$$(g(f(\varepsilon)))^{(n)}=\sum_{\substack{k=(k_1,\cdots,k_n)\in\mathbb{N}^n\\ \sum_{j=1}^n jk_j=n}} d_k g^{(\sum_{j=1}^n k_j)}(f(\varepsilon))\prod_{j=1}^n(f^{(j)}(\varepsilon))^{k_j},$$

where d_k are integer numbers depending only on k. Notice that when $k_n=1$ one has $k_1=\cdots=k_{n-1}=0$ and $d_k=1$.

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